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CONTENTS

On Integrability of the Sub-Riemannian Geodesic Flow for Goursat Distribution

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Consider the following optimal control problem:

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \qquad q \in \mathbb{R}^n, \quad u \in \mathbb{R}^2,$$

where $q = (x_1, x_2, \dots, x_n)^T$, $f_1 = (1, 0, -x_2, -x_3, \dots, -x_{n-1})^T$, $f_2 = (0, 1, 0, 0, \dots, 0)^T$, boundary conditions:

$$q(0) = q_0, \ q(t_1) = q_1,$$

quality functional:

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to min,$$

where the point $q \in \mathbb{R}^n$ determines the state of the system, $u = (u_1, u_2)$ is a control, t_1 being fixed.

Notice that f_1, f_2 can be chosen just as they are up to any diffeomorphism. This is how the commutators of f_1 and f_2 look like:

$$f_3 = \frac{\partial f_2}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_2 = [f_1, f_2] = (0, 0, 1, 0, \dots, 0)^T,$$

$$f_4 = \frac{\partial f_3}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_3 = [f_1, f_3] = (0, 0, 0, 1, \dots, 0)^T,$$

$$\dots$$

$$f_n = \frac{\partial f_{n-1}}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_{n-1} = [f_1, f_{n-1}] = (0, 0, \dots, 0, 1)^T.$$

Nilpotent Lie algebra is generated by f_1, f_2:

$$Lie(f_1, f_2) = span(f_1, f_2, \dots, f_n),$$

multiplication table being of the form:

$$[f_1, f_2] = f_3, \ [f_1, f_3] = f_4, \ \dots \ , \ [f_1, f_{n-1}] = f_n.$$

All the others are equal to zero. These relations define the so-called Goursat distribution ([1], [2]).

Using the Pontryagin maximum principle, one can construct the Hamiltonian

$$H(q, p, u) = \frac{u_1^2 + u_2^2}{2} = \frac{\dot{x_1}^2 + \dot{x_2}^2}{2} = \frac{\langle p, f_1 \rangle^2 + \langle p, f_2 \rangle^2}{2},$$

thus obtaining the following system:

$$\begin{cases} \dot{x_1} = p_1 - x_2p_3 - \ldots - x_{n-1}p_n, \\ \dot{x_2} = p_2, \\ \dot{x_3} = -x_2\dot{x_1}, \\ \ldots \\ \dot{x_n} = -x_{n-1}\dot{x_1}, \\ \dot{p_1} = 0, \\ \dot{p_2} = p_3\dot{x_1}, \\ \ldots \\ \dot{p_{n-1}} = p_n\dot{x_1}, \\ \dot{p_n} = 0. \end{cases} \tag{1}$$

Let us introduce the new coordinates by the following way:

$$\begin{cases} P_1 = p_1 - x_2p_3 - x_3p_4 - \ldots - x_{n-1}p_n, \\ P_n = p_n, \\ P_{n-1} = p_{n-1} - P_nx_1, \\ P_{n-2} = p_{n-2} - P_{n-1}x_1 - P_n\frac{x_1^2}{2!}, \\ \ldots \\ P_3 = p_3 - P_4x_1 - P_5\frac{x_1^2}{2!} - \ldots - P_n\frac{x_1^{n-3}}{(n-3)!}, \\ P_2 = p_2. \end{cases}$$

The following theorem holds.

Theorem. (1) *is the completely integrable system (in the Liouville sense). The whole set of the first integrals is as follows:*

$$\begin{cases} F_n = P_n, \\ F_{n-1} = P_{n-1}, \\ \ldots \\ F_3 = P_3, \\ F_2 = P_2 - P_3x_1 - \ldots - P_n\frac{x_1^{n-2}}{(n-2)!}, \\ F_1 = H = \frac{1}{2}(P_1^2 + P_2^2), \end{cases}$$

Thus one can consider the following "moment map":

$$\Phi\colon (x, P) \to \begin{pmatrix} F_n \\ \ldots \\ F_1 \end{pmatrix}.$$

The primary aim here is to study critical points of this mapping and its properties. That's what we are keep working on.

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Nilpotent Sub-Riemannian Problem on the Engel Group

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The following sub-Riemannian problem is considered:

$$\begin{aligned}
&\dot{q} = u_1 X_1 + u_2 X_2, \quad q = (x, y, z, v)^T \in M = \mathbf{R}^4, \quad (u_1, u_2) \in \mathbf{R}^2, \\
&X_1 = \left(1, 0, -\frac{y}{2}, 0\right)^T, \quad X_2 = \left(0, 1, \frac{x}{2}, \frac{x^2 + y^2}{2}\right)^T, \\
&q(0) = q_0 = (0, 0, 0, 0)^T, \quad q(t_1) = q_1, \\
&l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .
\end{aligned}$$

It arises as a nilpotent approximation to nonholonomic systems in four-dimensional space with two-dimensional control, for instance for the system describing motion of a mobile robot with a trailer on a plane.

Vector fields at the controls X_1, X_2 generate four-dimensional nilpotent Lie algebra called the Engel algebra [1]. X_1, X_2, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$ are basis left invariant fields on the Engel group M [2]. The system is completely controllable by Rashevskii–Chow theorem [3]. Existence of optimal solutions is implied by Filippov theorem.

Pontryagin's maximum principle has been applied. Projections of abnormal extremals on the plane XY are straight lines. Family of all normal extremals is parametrized by the phase cylinder of pendulum

$$C = T^*_{q_0} M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1; c, \alpha \in \mathbf{R}\},$$

where H is the Hamiltonian function.

Adjoint subsystem of the Hamiltonian system is reduced to the equation of pendulum:

$$\ddot{\theta} = -\alpha \sin\theta, \qquad \alpha = \text{const} .$$

The cylinder C has the stratification by value of the energy integral. Every subset of the cylinder corresponds to the particular type of trajectories of the pendulum. Hamiltonian system has been integrated in every case [4], thus exponential mapping is defined as:

$$\text{Exp}\colon N \to M, \qquad N = C \times \mathbf{R}_+ .$$

Discrete symmetries of exponential mapping have been considered in order to find the first Maxwell time which gives upper bound for the cut time (i. e., the time of loss of global optimality) along extremal trajectories:

$$t_{\text{cut}}(\lambda) \le t_{\text{MAX}^1}(\lambda).$$

Moreover, the first conjugate time (i. e., the time of loss of local optimality) along the trajectories has been investigated [5]. The function that gives the upper bound of the cut time provides the lower bound of the first conjugate time:

$$t_{\text{MAX}^1}(\lambda) \le t_{\text{conj}^1}(\lambda).$$

So the first Maxwell time defines the decomposition of the preimage and the image of the exponential mapping into corresponding subdomains. Hadamard theorem about global diffeomorphism has been applied to prove that restriction of the exponential mapping for these subdomains is a diffeomorphism. Finally the following theorem has been proved.

Theorem. *For any* $\lambda \in C$

$$t_{\mathrm{cut}}(\lambda) = t_{\mathrm{MAX}^1}(\lambda)$$

On the basis of the results obtained, a software for numerical computation of a global solution to the sub-Riemannian problem on the Engel group has been developed.

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Polynomial Integrals of the Geodesics Equations in Two-Dimensional Case

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Let M^2 be two-dimensional surface with the Riemannian metric

$$ds^2 = g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2. \tag{1}$$

Geodesics equations of a given metric can be treated as a system of Euler-Lagrange equations

$$\frac{d}{dt}L_{\dot{x}} - L_x = 0, \qquad \frac{d}{dt}L_{\dot{y}} - L_y = 0 \tag{2}$$

with the Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}g_{11}(x,y)\dot{x}^2 + g_{12}(x,y)\dot{x}\dot{y} + \frac{1}{2}g_{22}(x,y)\dot{y}^2. \tag{3}$$

Geodesic flow of the metric (1) is Liouville integrable, if it possesses a smooth first integral F functionally independent of the Lagrangian (3). In the present work we consider the problem of the existence of the polynomial integral of the system (2), (3) of the first degree

$$F_1 = b_0(x,y)\dot{x} + b_1(x,y)\dot{y}, \tag{4}$$

the second degree

$$F_2 = b_0(x,y)\dot{x}^2 + 2b_1(x,y)\dot{x}\dot{y} + b_2(x,y)\dot{y}^2 \tag{5}$$

and the third degree

$$F_3 = b_0(x,y)\dot{x}^3 + 3b_1(x,y)\dot{x}^2\dot{y} + 3b_2(x,y)\dot{x}\dot{y}^2 + b_3(x,y)\dot{y}^3. \qquad (6)$$

For an integral (4), (5) or (6) the existence conditions are obtained as the compatibility conditions of an overdetermined system of linear homogeneous first-order equations in the functions $b_i(x,y)$. Here these conditions are expressed in terms of the invariants

$$I_1(x,y) = \frac{J_1}{j_0 J_0^3}, \qquad I_2(x,y) = \frac{J_2}{j_0 J_0^2} \qquad (7)$$

of the equivalence transformations of the family of equations (2), (3) defined by

$$\tilde{t} = k(t+t_0), \qquad \tilde{x} = \varphi(x,y), \qquad \tilde{y} = \psi(x,y), \qquad k, t_0 = \text{const.}$$

In (7) the value J_0 up to a constant multiplier coincides with the main (scalar) curvature K of the surface M^2,

$$j_0 = g_{11}g_{22} - g_{12}^2, \qquad J_1 = g_{22}J_{0x}^2 - 2g_{12}J_{0x}J_{0y} + g_{11}J_{0y}^2.$$

All results on the integrals (4)–(6) are obtained in assumption of the non-degeneracy of the surface M^2. It means, when the conditions

$$j_0 \neq 0, \qquad J_0 \neq 0, \qquad J_1 \neq 0 \qquad (8)$$

hold. The geometrical sense of the first two conditions (8) is obvious (non-degeneracy of the matrix $g_{ij}(x,y)$ and nonzero curvature of the surface). The sense of the third condition (8) is not so evident. The question is what properties has the degenerate surface M^2 with the curvature K, which satisfies the relation

$$g_{22}(x,y)\left(\frac{\partial K}{\partial x}\right)^2 - 2g_{12}(x,y)\frac{\partial K}{\partial x}\frac{\partial K}{\partial y} + g_{11}(x,y)\left(\frac{\partial K}{\partial y}\right)^2 = 0. \qquad (9)$$

Heat Kernel Asymptotics at the Cut Locus on Riemannian and Sub-Riemannian Manifolds

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In this talk I will discuss the asymptotics of the heat kernel $p_t(x,y)$ on a Riemannian or sub-Riemannian manifold. We will consider the small time asymptotics, both off-diagonal and at the cut locus, showing how the asymptotic of $p_t(x,y)$ behave depending on whether (and how much) y is conjugate to x. Our results are obtained by extending an idea of Molchanov from the Riemannian to the sub-Riemannian case, and some details we get appear to be new even in the Riemannian context.

If time permits I will discuss how these techniques let us to identify the possible asymptotics for the heat kernel at the cut locus for a generic Riemannian manifolds (of dimension less or equal than 5). This is a consequence of the fact that, among the stable singularities of Lagrangian maps appearing in the classification of Arnold, only two of them can appear as "optimal", i.e. along minimizing geodesics.

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Geometric and Analytic Properties of Carnot–Carathéodory Spaces under Minimal Smoothness

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We describe geometric and analytical results in the theory of non-holonomic spaces under minimal smoothness, which we define following works [1, 2].

Definition. Fix a connected Riemannian C^∞-manifold $\mathbb{M}$ of topological dimension N. The manifold $\mathbb{M}$ is called the *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq H_2\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $x_0 \in \mathbb{M}$ has a neighborhood $U(x_0) \subset \mathbb{M}$ equipped with a collection of C^1-smooth vector fields $X_1, \ldots, X_N$ enjoying the following two properties:

(1) At every point $x \in U(x_0)$ we have a subspace

$$H_i\mathbb{M}(x) = H_i(x) = \operatorname{span}\{X_1(x), \ldots, X_{\dim H_i}(x)\} \subset T_x\mathbb{M}$$

of the dimension $\dim H_i$ independent of x, $i = 1, \ldots, M$.

(2) The inclusion $[H_i, H_j] \subset H_{i+j}$ holds for $i + j \leq M$.

Moreover, the Carnot–Carathéodory space is called the *Carnot manifold* if the following third condition holds:

(3) $H_{j+1} = \operatorname{span}\{H_j, [H_1, H_j], \ldots, [H_k, H_{j+1-k}]\}$, where $k = \lfloor \frac{j+1}{2} \rfloor$ for $j = 1, \ldots, M-1$.

Since it is not a priori known whether Carnot manifolds carry Carnot–Carathéodory metric, the Nagel–Stein–Wainger "Box" metric $d_\infty(x, y)$ is used instead in their study. Using results on fine properties of Carnot–Carathéodory spaces [2] we show that Carnot–Carathéodory metric is well-defined proving an analogue of Carathéodory– Rashevskiĭ–Chow theorem.

Theorem [3]**.** 1) *For every point $g \in \mathbb{M}$ there is a neighborhood U and $C > 0$ such that every point $x \in U$ can be represented as*

$$x = \exp(a_L X_{i_L}) \circ \ldots \circ \exp(a_1 X_{i_1})(g)$$

with $i_k \in \{1, \ldots, \dim H_1\}$ and $|a_k| \leq C d_\infty(x, g)$ for $k = 1, \ldots, L$. Here $L = L(\mathbb{M})$ does not depend on the points g, x.

2) *In a connected Carnot manifold any two points can be joined by an absolutely continuous curve consisting of finitely many segments of integral lines of vector fields $X_1, \ldots, X_{\dim H_1}$.*

This result in turn was utilized in [4] to prove local equivalence of "Box" metric d_∞ and Carnot–Carathéodory metric d_{cc} which immediately implies that:

• locally there is $C > 0$ such that $B(x, C^{-1}r) \subset \mathrm{Box}(x, r) \subset B(x, Cr)$;

• the Hausdorff dimension of $\mathbb{M}$ is $\nu = \sum\limits_{k=1}^{N} \deg X_k$;

• the Hausdorff measure $\mathcal{H}^\nu$ is locally doubling.

As an application of these results to a theory of Sobolev spaces we obtain the Poincaré inequality for John domains in Carnot manifolds.

Theorem [5]. *Let $x_0 \in \mathbb{M}$ and $1 \leq p < \infty$. There are $C_p > 0$ and $r_0 > 0$ such that for every John domain $\Omega \subset B(x_0, r_0)$ of class $J(a, b)$ and every $f \in C^\infty(\overline{\Omega})$ we have*

$$\|f - f_\Omega\|_{L_p(\Omega)} \leq (\tfrac{b}{a})^\nu \mathrm{diam}(\Omega) \|(X_1 f, \dots, X_{\dim H_1} f)\|_{L_p(\Omega)}$$

where $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f$ and ν is the Hausdorff dimension of $\mathbb{M}$.

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Almost-Riemannian Geometry of the Two-Sphere

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Consider the two following vector fields on S^2:

$$f_1(x) = x \times e_2, \qquad f_2(x) = x \times \sqrt{1 - a^2} e_1, \qquad x \in \mathbb{R}^3, \qquad |x| = 1,$$

where e_i, $i = 1, 2, 3$ is the standard basis of $\mathbb{R}^3$ and $a \in (0, 1)$ is a parameter. These vector fields correspond to rotations around axis OX_2 and OX_1.

Vector fields f_1 and f_2 span a non-constant rank distribution Δ:

$$\Delta_x = \mathrm{span}\{f_1(x), f_2(x)\}.$$

It's easy to see that rank $\Delta_x = 2$ almost everywhere, except for the equator, where f_1 and f_2 are collinear. The equator $\{x \in S^2 \colon x_3 = 0\}$ is called the singular set. Nevertheless any two points can be joined by a horizontal curve, which follows from the fact that

$$\Delta_x + [\Delta, \Delta]_x = T_x S^2.$$

Assume that there is a scalar product $g(\cdot,\cdot)$ on Δ for which the two vector fields f_1 and f_2 are othonormal:

$$g(f_i, f_j) = \delta_{ij}, \qquad i, j = 1, 2.$$

A triple (S^2, Δ, g) is called an almost-Riemannian sphere. In fact, everywhere except the singular set metric g is just a Riemannian metric on the sphere.

In the talk the problem of finding minimal curves of this structure will be discussed. This problem can be formulated as an optimal control problem:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x),$$

$$x, \omega \in \mathbb{R}^3, \qquad |x| = 1,$$

$$(u_1, u_2) \in \mathbb{R}^2, \qquad a \in (0, 1),$$

$$x(0) = \gamma_0, \qquad x(T) = x_T,$$

$$\int_0^T \sqrt{u_1^2 + u_2^2}\, dt \to \min.$$

We'll give a full parameterization of the geodesics and show how this problem is connected with the sub-Riemannian problems on SO(3). We'll also give description of Maxwell sets and bounds on the cut time.

Optimal R&D Policy in a Model Based on Exhaustible Resources

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We study an infinite-horizon optimal control problem arising from an endogenous growth model in which both production and research require an exhaustible resource. The model is a development of the earlier considered problem [2] (going back to Jones [4, 5]) of optimal extraction and use of a finite stock of some resource:

$$Y(t) = A(t)^{\kappa} L^Y(t)^{\alpha} R_1(t)^{1-\alpha} \qquad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad \kappa > 0, \tag{1}$$

$$\dot{A}(t) = A(t)^{\theta} L^A(t)^{\eta} R_2(t)^{\beta} \qquad \text{where} \quad \eta \in (0,1], \quad \beta \in [0, 1-\eta], \quad \text{and} \quad \theta \in (0,1]. \tag{2}$$

Here $Y(t)$ is the output produced at time t and $A(t)$ is the current knowledge stock. The resource is divided between production ($R_1(t)$) and research ($R_2(t)$). The total amount of the extracted resource cannot exceed the initial supply $S_0 > 0$ of the resource:

$$\int_0^{\infty} \big[R_1(t) + R_2(t)\big]\, dt \le S_0. \tag{3}$$

The population (total labor supply) is fixed at a certain level $L > 0$. Part of the labor $L^Y(t)$ is employed in production, while the other part $L^A(t) \in [0, L)$ is allocated to research:

$$L^A(t) + L^Y(t) \equiv L. \tag{4}$$

We consider a discounted logarithmic utility function of the output as a measure of *welfare*:

$$J_0(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^\infty e^{-\rho t} \ln Y(t)\, dt \to \max, \tag{5}$$

where $\rho > 0$ is a subjective discount rate.

As our study has shown [2], in the nonexceptional case of $(1-\beta)\theta < 1$, the labor and resource allocated (optimally) to research gradually decrease and ultimately vanish. Accordingly, the expansion of the knowledge stock is limited and stops or virtually stops at some moment and the output depletes to zero in the long run. However, experience suggests that there may occur jumps (transitions) from one technological trajectory to another. So we develop the above model further in order to take account of the possibility of such a jump. Namely, we assume that the moment of a jump T is a random variable such that

$$P(T < t + \Delta t \mid t \le T) = \nu L^A(t)\Delta t + o(\Delta t), \qquad o(\Delta t)/\Delta t \to 0 \quad \text{as} \quad \Delta t \to 0. \tag{6}$$

Then the probability density function for the random variable T is

$$\nu L^A(t) e^{-\nu \mathfrak{L}(t)}, \qquad \text{where} \qquad \mathfrak{L}(t) = \int_0^t L^A(s)\, ds \quad \text{for} \quad 0 \le t < \infty,$$

provided that $\mathfrak{L}(\infty) = \int_0^\infty L^A(s)\, ds = \infty$. If $\mathfrak{L}(\infty) < \infty$, then there is a positive probability that the jump will not occur at all, i.e. $p(T = \infty) = e^{-\nu \mathfrak{L}(\infty)} > 0$.

It is important to note that the process described by relations (1)–(4) is now of finite duration with probability $1 - e^{-\nu \mathfrak{L}(\infty)}$. Hence the integral in (5) must be taken only over the interval $[0, T]$ rather than over $[0, \infty)$. However, some estimate of the knowledge stock accumulated by the moment $t = T$ should also be taken into account because the accumulated knowledge $A(T)$ augments the productivity of the production means and hence increases the welfare on the remaining time interval $[T, \infty)$. Thus, we come to the following functional measuring the welfare:

$$J_T(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^T e^{-\rho t} \ln Y(t)\, dt + e^{-\rho T} V(A(T)).$$

To determine the value of the accumulated knowledge $A(T)$, we consider an auxiliary simple optimization problem on the interval $[T, \infty)$, which yields

$$V(A(T)) = C + \frac{\kappa}{\rho} \ln A(T),$$

where $C = C(\rho, \kappa, L)$ is a constant.

In this situation it is natural to aim at maximizing the expectation of the utility functional $J_T(A(\cdot), L^A(\cdot), R_1(\cdot))$ considered as a function of the random variable T. After some transformations, we reduce the problem to an equivalent infinite-horizon optimal control problem. Using standard results, we show the existence of an optimal control in the resulting problem. Then we apply the recent version of the Pontryagin Maximum Principle [1] (see also [3]) and analyze the solutions of the arising Hamiltonian system of the PMP. In particular, it is interesting to compare the behavior of optimal controls in this problem with that in problem (1)–(5).

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Classification of Binary Forms with Control Parameter

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The aim of the talk is to classify binary forms, whose coefficients depend on control parameter, with respect to the action of some pseudogroup. We solve this problem in two steps. Firstly, we consider the action of our pseudogroup on the infinite prolongation of the differential Euler equation and find differential invariant algebra of this action. Secondly, using methods from geometric theory of differential equations, we prove that three dependencies between basic differential invariants and their invariant derivatives uniquely define the equivalent class of binary forms with control parameter.

Let us consider the space $V_n(u)$ of binary forms, whose coefficients depend on the control parameter:

$$f(x,y;u)=\sum_{i=0}^{n} a_i(u)x^i y^{n-i}, \quad \text{where } a_i \text{ are holomorphic functions.}$$

The pseudogroup $G := \mathrm{SL}_2 \leftthreetimes (\mathcal{F}(u)\times \mathrm{T}(u))$ acts on the space $V_n(u)$ in the following way:

1) “semisimple part” SL_2 acts by linear transformations of the coordinates (x,y):

$$\mathrm{SL}_2 \ni A\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A^{-1}\begin{pmatrix} x \\ y \end{pmatrix};$$

2) “functional part” $\mathcal{F}(u)$ acts by holomorphic transformations of the control parameter: $u\mapsto \varphi(u)$;

3) “torus” $\mathrm{T}(u)$ acts by multiplications on the holomorphic functions on the control parameter: $f\mapsto \lambda(u)f$.

Consider space $\mathbb{C}^3$ with coordinates (x,y,u) and k-jet space J^k of functions on it (all necessary definitions and facts can be found in [1]). Denote by $(x,y,u,h,h_x,h_y,h_u,\ldots)$ the coordinates in k-jet space.

Binary forms with control parameter can be considered as solutions of the *Euler differential equation*

$$\mathcal{E} := \{xh_x + yh_y = nh\} \subset J^1$$

(see also [2]). The action of the pseudogroup G on 0-jet space J^0 prolongs to the action on all prolongations $\mathcal{E}^{(k-1)} \subset J^k$ (see [1]).

Definition 1. *Differential invariant of the action of pseudogroup G of order k* is G-invariant function on manifold $\mathcal{E}^{(k-1)}$, which is polynomial in derivatives h_σ, h^{-1} and $(h_x h_{yu} - h_y h_{xu})^{-1}$ (see Theorem 1).

Remark. Function $h_x h_{yu} - h_y h_{xu}$ is "total Poisson bracket" $\{h, h_u\}$. Hence this function is a differential semi–invariant of pseudogroup G (see [3]).

Definition 2. *Invariant derivative* is a combination of total derivatives, which commutes with the action of group G.

Theorem 1. *Differential invariant algebra of the action of pseudogroup G on the manifold $\mathcal{E}^{(\infty)}$ is freely generated by differential invariant*

$$H := \frac{h_{xx}h_{yy} - h_{xy}^2}{h^2}$$

of order 2 and by invariant derivatives

$$\nabla_1 := \frac{h_y}{h} D_x - \frac{h_x}{h} D_y \quad \text{and} \quad \nabla_2 := \frac{h^2}{h_x h_{yu} - h_y h_{xu}} \cdot D_u$$

(where D_x, D_y, D_u are total derivative operators with respect to variables x, y, u correspondingly).

Definition 3. Binary form $f \in V_n(u)$ is said to be *regular*, if the restrictions of the invariants H, H_1 and H_2 on form f are functionally independent in points of some domain $\Omega \subset \mathbb{C}^3$ (here indexes denote the corresponding invariant derivatives ∇_1 and ∇_2).

Consider the regular binary form f. Then the restrictions of invariants H_{11}, H_{12} and H_{22} on form f can be extended through the restrictions of the invariants H, H_1 and H_2 on f:

$$H_{11} = A(H, H_1, H_2), \;\; H_{12} = B(H, H_1, H_2), \;\; H_{22} = C(H, H_1, H_2).$$

The triple (A, B, C) is said to be *triple of dependencies* of form f.

Theorem 2. *Two regular binary forms f and $\widetilde{f}$ with control parameters are G-equivalent iff the triples of dependencies coincide:*

$$(A, B, C) = (\widetilde{A}, \widetilde{B}, \widetilde{C}).$$

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Local Conformal Flatness of Left-Invariant 3D Contact Structures

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In this talk I want to address the problem of finding the locally flat left-invariant contact structures on a three dimensional Lie Group up to conformal transformations, that is I will determine the ones locally conformally equivalent to the Heisenberg algebra $\mathbb{H}_3$. In particular I will show how to build the Fefferman metric associated to a generic three dimensional contact structure (not necessarily left-invariant) and by means of this construction I will give the explicit formula for the (unique) conformal invariant associated to such a structure. Next, specializing the study to the left-invariant case, I will give a complete list of the locally conformally flat structures which may appear and I will find the explicit form of the maps $\varphi\colon M \to \mathbb{R}$ which flatten our structures, and I will show that they are essentially (i.e. up to multiplication by a constant) unique.

Theorem. *Let (M, Δ, g) be a left-invariant 3D contact structure. Then it is locally conformally flat if and only if its canonical frame satisfies one of the following*

$$i)\ \begin{cases} [f_2, f_1] &= f_0 + c_{12}^2 f_2, \\ [f_1, f_0] &= \frac{2}{9}\left(c_{12}^2\right)^2 f_2, \\ [f_2, f_0] &= 0. \end{cases} \qquad ii)\ \begin{cases} [f_2, f_1] &= f_0 + c_{12}^1 f_1, \\ [f_1, f_2] &= 0, \\ [f_2, f_0] &= -\frac{2}{9}\left(c_{12}^1\right)^2 f_2. \end{cases}$$

or

$$iii)\ \begin{cases} [f_2, f_1] &= f_0, \\ [f_1, f_0] &= \kappa f_2, \\ [f_2, f_0] &= -\kappa f_1, \qquad \kappa < 0. \end{cases}$$

Where κ is the curvature of the structure.

Open question 1. Give a complete classification (i.e. not just the locally conformally flat ones) of left-invariant three dimesional contact structures, up to real rescalings.

Open question 2. Give satisfactory criteria to determine whether a given three dimesional contact structure (not necessarily left-invariant) is locally conformally flat or not.

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Maxwell Strata and Conjugate Points in Sub-Riemannian Problem on Group SH(2)

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Sub-Riemannian geometry has experienced resurgence of interest and extensive research for past several decades. It has emerged as an extremely rich framework with a unique character seeking applications in various fields of pure and applied mathematics such as classical and quantum mechanics, control theory, geometric analysis, stochastic calculus and evolution equations. The renewed interest is also attributed to the fact that sub-Riemannian geometry has given entirely new and richer perspective to some older problems such as image inpainting, neurophysiology of vision and quantum control [1]. Consequently, research in sub-Riemannian problems via geometric control methods on various Lie groups such as the Heisenberg group, S^3, SL(2), SU(2), SE(2), Engel group etc. has been paticularly popular for two decades now. From control theory perspective, sub-Riemannian geometry models optimal control problems for nonholonomic systems such as motion planning and control of robots, falling cats, parking of cars, rolling of bodies on plane without sliding, satellites, vision, quantum phases and even finance. Magnificence of sub-Riemannian geometry as an optimal control framework drew our attention to the sub-Riemannian problem on the group of motions of pseudo Euclidean plane. The pseudo Euclidean plane F_1^{1+1} is $(1+1)$-dimensional space defined over field of real numbers $\mathbb{R}$ and endowed with a non-degenerate indefinite quadratic form q:

$$q(x) = x_1^2 - x_2^2.$$

The motions of pseudo Euclidean plane are distance and orientation preserving maps of the points in the plane. The motions describe the hyperbolic roto-translations of the pseudo Euclidean plane and form a 3-dimensional Lie group known as special hyperbolic group SH(2) [2]. The driftless control system on SH(2) is described as follows:

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \qquad q \in M = \mathrm{SH}(2), \qquad (u_1, u_2) \in \mathbb{R}^2. \tag{1}$$

Here, (1) is the control system with bounded inputs u_i and control distribution $\Delta = \mathrm{span}\{f_1, f_2\}$. The vector fields f_i satisfy the Lie bracket relations:

$$[f_2, f_1] = f_0, \qquad [f_1, f_0] = 0, \qquad [f_2, f_0] = f_1.$$

The sub-Riemannian problem on control system (1) is defined as:

$$q(0) = Id, \qquad q(t_1) = q_1, \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{3}$$

In (2), $q(0)$ and $q(t_1)$ represent the initial and the final states whereas l (3) is the sub-Riemannian distance (length functional) to be minimized. In coordinates $q = (x, y, z)$, the control system (1) is given as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cosh z \\ \sinh z \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2. \tag{4}$$

We applied the Pontryagin Maximum Principle (PMP) on (1)–(3) to calculate the extremal controls $\tilde{u}(t)$ and the extremal trajectories. Since the problem is 3D contact, there are no nontrivial abnormal trajectories. A change of coordinates in the vertical subsystem of the normal Hamiltonian system transforms it into a mathematical pendulum. The phase cylinder C of the pendulum is decomposed into five connected subsets $C_i \quad i = 1, \dots, 5$ depending upon the energy of the pendulum. Suitable elliptic coordinates i.e. reparametrized energy k and reparametrized time φ are introduced on each C_i and such that the flow of the vertical subsystem is rectified. Computation of the Hamiltonian flow then follows from integration of vertical and horizontal subsystem and the resulting extremal trajectories are parametrized by Jacobi elliptic functions. Further analysis/simulations reveal the qualitative nature of extremal trajectories.

Parametrization of extremal trajectories is followed by second order optimality analysis based on description of Maxwell strata and conjugate loci. Since the vertical subsystem is a mathematical pendulum, it admits reflection symmetries in the phase portrait which are used to obtain complete description of Maxwell strata. The fixed points of the extremals λ in the preimage and the multiple points in the image of exponential mapping are used to obtain complete description of the Maxwell strata and compute the first Maxwell time t_1^{MAX} for $\lambda \in C_i$, $i = 1, \dots, 5$. On the basis of Maxwell strata and Maxwell time, we obtain a global upper bound on cut time in the sub-Riemannian problem on SH(2) which happens to be the first Maxwell time t_1^{MAX}. We then turn to the problem of characterizing the conjugate points. Computation and simplification of Jacobian for $\lambda \in C_1 \cup C_2$ reveals a rather unexpected symmetry with respect to bounds of conjugate times in these cases which hasn't been observed in corresponding analysis in sub-Riemannian problem on SE(2) [3], Engel group [5] and Euler Elasticae problem [4]. It turns out that the first conjugate time $t_1^{C_1}$ for $\lambda \in C_1$ is bounded as $4K(k) \leq t_1^{C_1} \leq 2p_1^1(k)$ where $p_1^1(k)$ is the first root of a function $f_1(p) = [\mathrm{cn}p\, \mathrm{E}(p) - \mathrm{sn}p\, \mathrm{dn}p]$. The function $f_1(p)$ and its roots shall be described in more detail in our upcoming journal paper on Maxwell Strata on SH(2). Similarly, for $\lambda \in C_2$ first conjugate time is bounded as $t_1^{C_2} = kt_1^{C_1}$. Thus globally the first conjugate time is greater or equal to the first Maxwell time. We conjecture that the cut time is equal to the first Maxwell time. This conjecture will be studied in a forthcoming work.

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Self-adjoint Commuting Differential Operators of Rank 2 and Their Deformations Given by the Soliton Equations

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In [1] and [2] I.M. Krichever and S.P. Novikov introduced a remarkable class of exact solutions of soliton equations — solutions of rank $l > 1$. In this article we study solutions of rank two of the following system

$$V_t = \frac{1}{4}(6VV_x + 6W_x + V_{xxx}), \quad W_t = \frac{1}{2}(-3VW_x - W_{xxx}). \tag{1}$$

This system is equivalent to the commutativity condition of the self-adjoint operator

$$L_4 = (\partial_x^2 + V(x,t))^2 + W(x,t)$$

and the skew-symmetric operator $\partial_t - \partial_x^3 - \frac{3}{2}V(x,t)\partial_x - \frac{3}{4}V_x(x,t)$. In this case "solutions of rank two" means that for every $t \in \mathbb{R}$ every operator commuting with L_4 has even order. It also means that the dimension of space of common eigenfunctions of commuting operators L_4 and L_{4g+2} is equal to two

$$dim_{\mathbb{C}} \{\psi \colon L_4\psi = z\psi, L_{4g+2}\psi = w\psi\} = 2$$

for generic eigenvalues (z, w). The set of eigenvalues $P = (z, w)$ forms hyperelliptic curve

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0.$$

This curve is called spectral.

There is a classification of commutative rings of ordinary differential operators of arbitrary rank obtained by Krichever [3] but in general case such operators are not found.

Krichever and Novikov [1] found operators of rank two corresponding to an elliptic spectral curve. Mokhov found operators of rank three corresponding to an elliptic spectral curve. In the case of spectral curves of genus 2, 3 and 4 it is known only examples of operators of rank greater than one.

Operators L_4 and L_{4g+2} of rank two corresponding to hyperelliptic spectral curves were studied in [4]. Operators $L_4 - z$, $L_{4g+2} - w$ have common right divisor $L_2 = \partial_x^2 - \chi_1(x,P)\partial_x - \chi_0(x,P)$:

$$L_4 - z = \tilde{L}_2 L_2, \quad L_{4g+2} - w = \tilde{L}_{4g} L_2.$$

Functions $\chi_0(x,P)$, $\chi_1(x,P)$ are rational functions on Γ, they satisfy the Krichever's equations. The operator L_4 is self-adjoint if and only if $\chi_1(x,P) = \chi_1(x,\sigma(P))$ [4]. If $g \geq 1$, then the following theorem holds [4].

Theorem 1. [4] *If L_4 is self-adjoint operator, then*

$$\chi_0 = -\frac{Q_{xx}}{2Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q_x}{Q},$$

where $Q = z^g + \alpha_{g-1}(x)z^{g-1} + \ldots + \alpha_0(x)$. Polynomial Q satisfies equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} + 2Q(2V_xQ_x + 4VQ_{xx} + Q_{xxxx}). \tag{4}$$

The main aim of this paper is as follows. We study dynamics of polynomial Q provided that V and W satisfy (1).

Theorem 2. *Suppose that potentials V and W of operator $L_4 = (\partial_x^2 + V(x,t))^2 + W(x,t)$ commuting with operator L_{4g+2} satisfy the system (1). Then polynomial Q satisfies the following equation $Q_t = \frac{1}{2}\left(-3VQ_x - Q_{xxx}\right)$.*

Remark 1. Similarly one can obtain the evolution equation on Q if in (2) one substitutes operator A by a skew-symmetric operator of order $2n+1$. For example, in case of $n = 2, 3$.

The following theorems are proved in [4] and [6].

Theorem 3. *The operator*

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + \alpha_3 g(g+1)x$$

commutes with an operator $L_{4g+2}^\sharp$ of order $4g+2$. The spectral curve is given by the equation $w^2 = F_{2g+1}(z)$, where F_{2g+1} is a polynomial of degree $2g+1$.

Theorem 4. *The operator*

$$L_4^\natural = (\partial_x^2 + \alpha_1 \cosh(x) + \alpha_0)^2 + \alpha_1 g(g+1)\cosh(x), \quad \alpha_1 \neq 0$$

commutes with an operator $L_{4g+2}^\natural$ of order $4g+2$. The spectral curve is given by the equation $w^2 = F_{2g+1}(z)$, where F_{2g+1} is a polynomial of degree $2g+1$.

The following theorems were proved in collaboration with E.I. Shamaev.

Theorem 5. *The operator $L_4^\sharp$ does not commute with any differential operator of odd order.*

Theorem 6. *The operator $L_4^\natural$ does not commute with any differential operator of odd order.*

Theorems 5 and 6 rigorously prove that $L_4^\natural$ from [4] and $L_4^\sharp$ from [6] are differential operators of rank two.

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Composition Operators on Sobolev Spaces in a Carnot Group

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Mainly we study mappings inducing composition operators on Sobolev spaces. In this talk we are going to present the basic notions regarding the problem under consideration. Moreover, we formulate our main result for isomorphic composition operators of Sobolev spaces on a Carnot Group. This talk is based on a joint work with Sergey Vodopyanov [2]. We develop and generalize ideas from the framework for $\mathbb{R}^n$, see [1].

A **Carnot group** $\mathbb{G}$ is a connected simply connected stratified nilpotent Lie group. This means that the Lie algebra $\mathfrak{g}$ of the group $\mathbb{G}$ admits a nilpotent stratification: $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, and $[V_1, V_j] = V_{j+1}$ for $j = 1, \dots, m-1$, whereas $[V_1, V_m] = \{0\}$. Let $X_1, \dots, X_n$ be vector fields constituting a basis of V_1.

Sobolev space $L^1_p(D)$ consist of locally integrable functions $f\colon D \to \mathbb{R}$ with weak derivatives $X_i f \in L^1_p(D)$, $i = 1, \dots n$. Let $\varphi\colon D \to D'$ is a measurable mapping and $L^1_q(D)$, $L^1_p(D')$ are Sobolev spaces on these domains. If a function $f \in L^1_p(D')$ is continuous then the composition $f \circ \varphi$ is well-defined almost everywhere on D. Assume that $f \circ \varphi \in L^1_q(D)$ and $\|f \circ \varphi \mid L^1_q(D)\| \leqslant K \|f \mid L^1_p(D')\|$ for all $f \in L^1_p(D') \cap C(D')$. Thus have just defined the **composition operator**:

$$L^1_p(D') \cap C(D') \ni f \mapsto \varphi^* f = f \circ \varphi \in L^1_q(D). \tag{1}$$

It is well known that operator (1) can be extended to the whole space $L^1_p(D')$ by the continuity.

Here we consider the case $p = q$ and the extension of φ^* is an isomorphism.

Theorem. *Let $p \geq 1$, $p \neq \nu$, and D, D' are domains on a Carnot group $\mathbb{G}$. Measurable mapping $\varphi\colon D \to D'$ induces an isomorphism of Sobolev spaces*

$$\varphi^*\colon L^1_p(D') \to L^1_p(D),$$

if and only if φ coincides almost everywhere with a quasi-isometric homeomorphism (w.r.t. Carnot Carathéodory distance) $\Phi\colon D \to \Phi(D)$ for which Sobolev spaces $L^1_p(\Phi(D))$ and $L^1_p(D')$ are equivalent.

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From Approximate Reachable Sets to Asymptotic Control Theory*

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The problem of time-optimal steering of an initial state of a dynamical system to a given manifold is typical for the optimal control theory. Optimal trajectory is to be found as the steepest descent in the direction of the gradient of the cost function. The level sets of the cost functions are boundaries of the reachable set of the system in respect to backward time. The direction of the gradient coincides with the normal to boundary of the reachable set.

Definition. The reachable set $\mathcal{D}(T)$ is the set of ends at time instant T of all admissible trajectories of the system starting at the given manifold at zero time.

It is remarkable, that for a wide class of linear systems of the form

$$\dot{x} = Ax + Bu, \quad |u| \leq 1,$$

where u is a control, reachable set $\mathcal{D}(T)$ equals asymptotically as $T \to \infty$ to the set $T\Omega$, where Ω is a fixed convex body, (here given manifold is the origin). More than that, the support function H_Ω, which defines Ω uniquely, has an explicit integral representation. Starting from this point, we can design a control using steepest descent in the normal direction to the boundary of approximate reachable sets $T\Omega$.

Analytically speaking this means that for a state vector x we have to solve the following equation

$$x = T\frac{\partial H_\Omega}{\partial p}(p)$$

with unknown time T and momentum $p = p(x)$. The control we describe takes the form $u(x) = -\mathrm{sign}\langle B, p(x)\rangle$.

Following this strategy, we can make a damping of a non-resonant system of linear oscillators in quasi-optimal time. More precisely,

Theorem 1. *Assume that system of oscillators is non-resonant. Let $T = T(x)$ be the motion time from the initial point x to the equilibrium under our control, and $\tau = \tau(x)$ be the minimum time. Then, as the $x \to \infty$ we have the asymptotic equality*

$$\tau(x)/T(x) = 1 + o(1).$$

These general arguments to a great extent are applicable to the problem of damping of a closed string

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + u\delta, \quad |u| \leq 1.$$

Here, $x \in [0, 2\pi]$ is the angle coordinate on a one-dimensional torus $\mathcal{T}$, t is time, δ is the Dirac δ-function. Particularly, we obtain the following result

Theorem 2. *It is possible to damp the string by a bounded load applied to a fixed point in finite time, if at the initial state*

$$f \in L_\infty, \quad \frac{\partial f}{\partial x} \in L_\infty, \quad \frac{\partial f}{\partial t} \in L_\infty.$$

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Ma–Trudinger–Wang Tensor, from PDE Regularity to Geometric Information

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The Ma–Trundinger–Wang (MTW) tensor was introduced in [4] to guarantee a regularity theory for the fully non linear Monge-Ampère Equation. In particular this PDE is satisfied by the solution of an optimal transport problem [5]. This tensor also leads to a regularity theory (TCP theory) for optimal transportation problem on a Riemannian manifold [3]. For example we can set in dimension 2 the following theorem :

Theorem. *The TCP condition (continuity of optimal transport map) holds if and only if* (M, g) *satisfies* **(MTW)** *(positivity of MTW tensor) and all its injectivity domains are convex.*

Cédric Villani conjecture that the convexity of injectivity domains is in fact a consequence of **(MTW)**. He makes a step in this direction [1]. Together with Alessio Figalli and Ludovic Rifford [2] we improved this result and prove the following "Boney M theorem:

Theorem. *Let* (M, g) *be a nonfocal Riemannian manifold satisfying* **(MTW)**. *Then all injectivity domains of* M *are convex.*

During this talk we will introduce the optimal transportion problem and the Ma–Trundinger–Wang tensor. We then review some applications of MTW tensor in order to make it less mysterious and prove that it contains many geometric informations. In particular we will explain why the MTW tensor can be seen as a curvature one. We will conclude with the convexity of injectivity domains for a non focal manifold.

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Geodesics and Topology of Horizontal-Path Spaces in Carnot Groups

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On a sub-Riemannian manifold it is interesting to study the topology of the space of horizontal curves joining two points (the nonholonomic loop space); by applying Morse theory we can relate its topology with the structure of geodesics (critical points of the energy). Precisely we study the case where the points are 'infinitesimally close', in order to get the properties depending on the local structure of the distribution and avoiding properties due to the topology of the manifold.

This means that we focus on local models of sub-Riemannian manifolds, namely Carnot groups. We find the structure of the geodesics joining the origin with so called vertical points, where the most typical behaviour of nonholonomic constraints appear. Moreover, even though the space of horizontal paths is contractible, we measure its complexity by looking how the topology of sublevels of the Energy change, in the spirit of Morse theory.

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Differential Invariants of Feedback Transformations for Quasi-Harmonic Oscillation Equations

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The classification problem for a control-parameter-dependent second-order differential equations is considered. The algebra of the differential invariants with respect to Lie pseudo-group of feedback transformations is calculated. The equivalence problem for a control-parameter-dependent quasi-harmonic oscillation equation is solved. Some canonical forms of this equation are constructed.

Consider the problems of equivalence and classification for the differential equation:

$$\frac{d^2y}{dx^2} + f(y,u) = 0, \tag{1}$$

with respect to the feedback transformations [1]:

$$\varphi\colon (x,y,u) \longmapsto (X(x,y), Y(x,y), U(u)), \tag{2}$$

where the function$f(y,u)$ is smooth. Here u is a scalar control parameter. We will call an equation of form (1)*control-parameter-dependent quasi-harmonic oscillator equation* (QHO).

Definition. Operator

$$\nabla = M\frac{d}{dy} + N\frac{d}{du} \tag{3}$$

is called *G-invariant differentiation* if it commutes with every element of any prolongation of Lie algebra $\mathcal{G}$, where M and N are the functions on the jet space.

Theorem. *Differential operators*

$$\nabla_1 = \frac{z}{z_y}\frac{d}{dy}, \tag{4}$$

$$\nabla_2 = \frac{z}{z_u}\frac{d}{du} \tag{5}$$

are G-invariant differentiations.

Theorem. *Functions*

$$J_{21} = \frac{z_{yy}z}{z_y^2}, \quad J_{22} = \frac{z_{yu}z}{z_y z_u}$$

form a complete set of the basic second-order differential invariants, i.e.any other second-order differential invariants are the functions of J_{21} and J_{22}.

Theorem. *Quasi-harmonic oscillation equation differential invariants algebra is generated by second-order differential invariants J_{21}, J_{22} and invariant differentiations ∇_1 and ∇_2. This algebra separates regular orbits.*

Let us call an equation $\mathcal{E}_f$ *regular*, if

$$dJ_{21}(f) \wedge dJ_{22}(f) \neq 0.$$

Here $J(f)$ is the value of the differential invariant J on the function $f = f(y,u)$.

Theorem. *Suppose that the functions f and g are real-analytical. Two regular equations $\mathcal{E}_f$ and $\mathcal{E}_g$ are locally G-equivalent if and only if the functions Φ_{if} and Φ_{ig} identically equal ($i = 1,2,3$) and 3-jets of the functions f and g belong to the same connection component.*

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Absense of Local Maxima for Optimal Control of Two-Level Quantum Systems

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The goal of optimal control for a quantum system whose evolution is governed by Schrodinger equation is to find controls which maximize target objective functional, such as quantum average of some observable. Often numerical methods are used to find optimal controls. This makes important the problem of analysis of the existence or absence of local maxima (traps) of the target functional, since their presence may hinder the numerical search from finding true global maxima. Significant progress in the analysis of trap was made in recent works by H. Rabitz, A.N. Pechen, D.J. Tannor, R. Wu, C. Brif, P. de Fouquieres, S.G. Schirmer and others [1–3]. However, systems without traps were not known. In the joint work with A.N. Pechen [4] we present the proof of the absence of local maxima for a wide range of target functionals for two-level quantum systems governed by Schrodinger equation.

Theorem. *For two-level quantum system with controlled evolution*

$$i\frac{d}{dt}U_t^f = [H_0 + f(t)V]U_t^f, \quad [H_0, V] \neq 0$$

all maxima of functionals $J_{i\to f}(f) = |\langle\psi_f|U_T^f|\psi_i\rangle|^2$, $J_O(f) = \mathrm{Tr}(U_T^f\rho_0 U_T^{f\dagger}O)$, $J_W(f) = |\mathrm{Tr}(U_T^f W^\dagger)|^2$ *are global that is, there are no local maxima.*

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Diffusion by Optimal Transport in the Heisenberg Group

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In this talk, we will consider the hypoelliptic diffusion, the "heat diffusion" of the subRiemannian Heisenberg group $\mathbb{H}$. We will show that in the Wasserstein space $\mathcal{P}_2(\mathbb{H})$, the space of probability measures with finite second moment, it is a curve driven by the gradient flow of the Boltzmann entropy, $\mathrm{Ent}\colon \mathcal{P}_2 \to \mathbb{R} \cup \{\infty\}$. Conversely any gradient flow curve of Ent satisfies the hypoelliptic heat equation.

This illustrates and completes the theory of Ambrosio, Gigli ans Savaré about the gradient flows of Ent on the Wasserstein spaces of some very general metric spaces.

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Metric Geometry of Carnot–Carathéodory Spaces and Its Applications

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We describe new fine properties of Carnot–Carathéodory spaces under minimal assumptions on smoothness of the basis vector fields. As a corollary, we discover new geometric properties of weighted Carnot-Caratheodory spaces. All these results are new even for a "smooth" case. They play crucial role in the development of the differentiability theory on sub-Riemannian structures (see, e.g., works by S. Vodopyanov [1, 2]), in the investigation of non-equiregular Carnot–Carathéodory spaces (see, e.g., work by S. Selivanova [3]) and imply many basic results of the theory of non-holonomic spaces (see, e.g., work by S. Basalaev and S. Vodopyanov [4]).

Definition see, e.g., [5, 2, 4, 6, 7]**.** Fix a connected Riemannian C^∞-manifold $\mathbb{M}$ of topological dimension N. The manifold $\mathbb{M}$ is called the *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of C^1-smooth vector fields $X_1, \ldots, X_N$ enjoying the following two properties.

(1) At every point $v \in U$ we have a subspace

$$H_i\mathbb{M}(v) = H_i(v) = \operatorname{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\} \subset T_v\mathbb{M}$$

of the dimension $\dim H_i$ independent of v, $i = 1, \ldots, M$.

(2) The inclusion $[H_i, H_j] \subset H_{i+j}$, $i + j \leq M$, holds.

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the *Carnot manifold*:

(3) $H_{j+1} = \operatorname{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$, where $k = \left[\frac{j+1}{2}\right]$, $H_0 = \{0\}$, $j = 1, \ldots, M-1$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold $\mathbb{M}$.

The main result is the following

Theorem [6, 7]**.** *Let $\mathbb{M}$ be a Carnot–Carathéodory space with $C^{1,\alpha}$-smooth basis vector fields, $\alpha \geq 0$ (if $\alpha = 0$ then the fields belong to the class C^1). Then for each point of $\mathbb{M}$, there exists a sufficiently small neighborhood $\mathcal{U} \Subset \mathbb{M}$ possessing the following property: for $u, v \in \mathcal{U}$, $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma}\colon [0,1] \to \mathbb{M}$ are absolutely continuous (in the classical sense) curves contained in $\operatorname{Box}(u, \varepsilon)$ such that*

$$\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t) X_i(\gamma(t)), \quad \gamma(0) = v, \quad \textit{and} \quad \dot{\widehat{\gamma}}(t) = \sum_{i=1}^{N} b_i(t) \widehat{X}_i^u(\gamma(t)), \quad \widehat{\gamma}(0) = v,$$

and each measurable function $b_i(t)$ meets the property

$$\int_0^1 |b_i(t)|\, dt < S\varepsilon^{\deg X_i}, \tag{1}$$

$S < \infty$, $i = 1, \ldots, N$, we have

$$\max\{d_\infty(w, \widehat{w}), d^u_\infty(w, \widehat{w})\} = \begin{cases} O(1) \cdot \varepsilon^{1+\frac{\alpha}{M}} \text{ if } \alpha > 0, \\ o(1) \cdot \varepsilon \text{ if } \alpha = 0, \end{cases} \tag{2}$$

with $O(1)$ and $o(1)$ to be uniform in $u \in \mathcal{U}$ and all collections of functions $\{b_i(t)\}_{i=1}^N$ with the property (1).

Remark [7]; see also [8]**.** For weighted Carnot–Carathéodory spaces, the estimate in (2) is $O(1) \cdot \varepsilon^{1+\frac{\alpha}{l_M}}$ for $\alpha > 0$, where l_M is the maximal weight [3, 7, 8].

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On an Infinite Horizon Problem of Bolza Type*

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The first necessary conditions of optimality for infinite-horizon control problems were proved [1] on the verge of 1950–60s by L.S. Pontryagin and his associates (for the problems with the right end fixed at infinity). Only later [2] was the Maximum Principle proved for a reasonably broad class of problems, and yet the transversality-type conditions at infinity were not provided. Thus, the Maximum Principle for infinite horizon was not complete, which means the set of prospective optimal solutions it determined had the cardinality of continuum.

The principal obstacle on the way to transversality conditions at infinity is the fact that it is necessary to find the asymptotic conditions on the adjoint equation (i.e., on the linear system), that

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would be satisfied by at least one but not by all of its solutions. It was first done in [3] for a system with linear dynamics and the free right-hand end through passing to a functional space that allowed to extend all necessary solutions to infinity in the unique way. For the adjoint variable, there was proved a formula that supplemented the Maximum Principle to make it a complete system. In the papers [4–6], a more general formula (the Aseev–Kryazhimskii formula) was proved for other certain classes of nonlinear control problems. It takes the form of an improper integral of a function, the summability of which on the whole half-line is provided by means of imposing the asymptotic conditions (similar to the dominating discount conditions) on the system.

Another way to decrease the number of prospective solutions of such an incomplete system of relations was proposed by Seierstad [7]. He considered a set of shortened problems, in each of which he obtained the adjoint variable in the form of a solution of the complete system of relations (the Maximum Principle system for shortened problem). Under sufficiently strong assumptions he made, the adjoint variable, obtained as a pointwise limit, satisfied the maximum principle. The author extended this approach onto the class of infinite horizon control problems with the free right-hand end to the case of at least when the optimality criterion is at least the uniformly weakly overtaking optimality [8]. In particular, the transversality condition obtained through this means may be represented in the form of an Aseev–Kryazhimskii-type formula.

The author plans to report on the application of this approach to the problem of Bolza type.

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Sub-Riemannian and Riemannian Structures on the Lie Algebroids

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A Lie Algebroid is generalization of Lie Algebra for orbitrary vector bundle over a some manyfold. An Affinor Structure is generalization of Almost Complex Structure preserving the some symplectic form for fixed regular 1-form with nontrivial radical of orbitrary dimension. Unlike a Symplectic or

Contact structure Afffinor Structure can be taken on any Lie Algebroid of any dimension, and the based 1-form can be degenerated having the radical of orbitrary dimension. These Affinor Structures generate a special Subriemannian and Riemannian structures on Lie Algebroids.

Laplacian Flow of G_2-Structures on $S^3 \times \mathbb{R}^4$

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A 7-dimensional smooth manifold M admits a G_2-structure if there is a reduction of the structure group of its frame bundle from $GL(7,\mathbb{R})$ to the group G_2, viewed as a subgroup of $SO(7,\mathbb{R})$. On a manifold with G_2-structure there exists a "non-degenerate" 3-form φ, which determines a Riemannian metric g_φ in a non-linear fashion. Let (M,φ) be a manifold with G_2-structure. If φ is parallel with respect to Levi-Civita connection of the metric g_φ, $\nabla\varphi = 0$, then (M,φ) is called G_2-manifold. Such manifolds are always Ricci-flat and have holonomy contained in G_2. The condition $\nabla\varphi = 0$ is equivallent to φ to be closed, $d\varphi = 0$, and co-closed, $\delta\varphi = 0$, form. It is very interesting to understand how we can get a parallel φ on a certain manifold with G_2-structure via the evolution of some specific quantities. I will tell about the flow $\frac{\partial\varphi(t)}{\partial t} = \Delta\varphi$ on a $S^3 \times \mathbb{R}^4$, where $\varphi(t)$ is a continuous family of G_2-structures defined on this space and $\Delta = d\delta + \delta d$ is a Hodge-Laplacian operator.

Regularity of Isometries of Sub-Riemannian Manifolds

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We consider manifolds equipped with Carnot-Caratheodory distances and discuss some methods to show smoothness of their isometries (i.e., their distance-preserving homeomorphisms). The arguments come from analysis on metric spaces, PDE, and the theory of locally compact groups. It will be important to consider the metric tangent spaces of subRiemannian manifolds, which are Carnot groups. We explain why isometries between Carnot groups are affine maps and also the fact that subRiemannian isometries, likewise the Riemannian ones, are uniquely determined by the horizontal differential at a point. The work is in collaboration with L. Capogna and A. Ottazzi.

How Many Geodesics Are There between Two Close Points on a Sub-Riemannian Manifold?

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Given a point q on a Riemannian manifold and a small enough neighborhood U of this point, then for every other point $p \in U$ there will be only one geodesic joining these two points entirely contained in U.

Moving to the *sub*-Riemannian case, the situation dramatically changes.

Consider for example, the standard Heisenberg group $\mathbb{R}^3$ with coordinates (x, y) (here y is the "vertical" coordinate). Then the number $\hat{\nu}(p)$ of geodesics joining the origin with the point $p = (x, y)$ is given by:

$$\hat{\nu}(p) = \frac{8\|y\|}{\pi\|x\|^2} + O(1) \qquad (1)$$

One should notice, for instance, that when the point is "vertical" ($x = 0$) there are infinitely many geodesics and when the point is "horizontal" ($y = 0$) there are finitely many (in fact just one).

On a general sub-Riemannian manifold, given a point q and *privileged coordinates* on a neighborhood U of q, one can consider the associated family of dilations:

$$\delta_\epsilon \colon U \to U, \quad \delta_\epsilon(q) = q.$$

When ϵ is very small, the geometry of this family approaches a limit geometry: the sub-Riemannian tangent space at q (a Carnot group).

Given another point $p \in U$, it is natural to ask for the number $\nu(\delta_\epsilon(p))$ of geodesics between q and $\delta_\epsilon(p)$ (i.e. when the two points get closer and closer, in the sub-Riemannian sense).

In this talk I will show how to relate the asymptotic for $\nu(\delta_\epsilon(p))$ to the count on the associated Carnot group (as performed in formula (1) above). I will show, for instance, that for the generic $p \in U$:

$$\lim_{\epsilon \to 0} \nu(\delta_\epsilon(p)) = \hat{\nu}(p)$$

and discuss related questions and applications.

This is joint work with L. Rizzi

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Hamiltonian Flow of Singular Trajectories

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The Pontryagin maximum principle reduces problems of optimal control to the study of Hamiltonian systems of ODEs with discontinuous right-hand side. Optimal synthesis is the set of solutions of this system with a fixed end (or initial) condition covering a certain region of the phase space in a unique way. Singular trajectories play key role in the construction of an optimal synthesis. These trajectories lie in the surface of discontinuity of the right-hand side of the Hamiltonian system.

On the report, recently proved theorem on Hamiltonian property of singular flow will be discussed. Namely, the set of singular trajectories of a fixed order forms a symplectic manifold, and the singular flow on it is Hamiltonian.

The result is constructive and makes it possible to apply the full spectrum of the theory of Hamiltonian systems to the study of singular trajectories. As an example of the use of this theorem I consider the control problem of magnetized Lagrange top in a changing magnetic field. It is proved that the flow of singular trajectories in this problem is completely integrable in the Liouville sense and is included in the flow of a smooth superintegrable Hamiltonian system in the ambient space. Direct study of this problem (without using the proposed technique) is seemed to be impossible because of the huge complexity of direct calculations.

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The Average Number of Connected Components of an Algebraic Hypersurface

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How many zeros of a random polynomial are real? M. Kac [2] tackled this question for a Gaussian ensemble of univariate polynomials (1943). Addressing the case of a real algebraic hypersurface in $\mathbb{R}P^n$, we discuss asymptotic estimates for the number (and relative position) of connected components. This addresses a random version of Hilbert's Sixteenth Problem.

The outcome depends on the definition of "random". We consider Gaussian ensembles that are invariant under an orthogonal change of coordinates. Following E. Kostlan [3] we parameterize this family of ensembles in terms of a generalized Fourier series of eigenfunctions of the spherical Laplacian. With some regularity assumptions on the choice of weights assigned to each eigenspace, we calculate the order of growth (as the degree d goes to infinity) of the average number of connected components. The order of growth turns out to be the same as the nth power of the average number of zeros on a one-dimensional sample slice:

$$\mathbb{E}b_0(X) = \Theta\left(\left[\mathbb{E}b_0(X \cap \mathbb{R}P^1)\right]^n\right), \quad \text{as } d \to \infty.$$

This relates the multivariate case to the classical problem of Kac.

The proof uses random matrix theory to prove an upper bound and harmonic analysis to prove a lower bound. (This is joint work with Yan V. Fyodorov and Antonio Lerario [1, 4].)

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Construction of Classical Metrics with Special Holonomies via Geometrical Flows

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Many well-known metrics with curvature restrictions (such as special holonomy) have form $\bar{g} = dt^2 + g(t)$, where the metric $g(t)$ is usually a deformed metric on some well-studied space, for example homogenous space. The deformed metric $g(t)$ depends on the functions of variable t and the curvature restrictions which are the equations on the Riemannian or Ricci tensors instead of being partial differential equations become a ordinary differential equations. A class of interesting metrics appears when $\bar{g}$ is a cone metric $dt^2 + t^2ds^2$. If one wants to get a flow that gives a constant curvature metric $\bar{g}$, he will be led to the flow

$$\frac{\partial}{\partial t} g = \sqrt{Ric - 4K}.$$

We call this flow the *Dirac flow*. Although the right-hand side of this flow is pseudodifferential operator of the first order the qualitative behavior of this flow is similar to the behavior of Ricci flow (at least for the simplest case of conformally round metric on S^3). This flow collapses the 3-dimensional sphere, such behavior at the origin $t = 0$ is so-called "nut"-type singularity. But some important metrics (e.g. Eguchi-Hanson metric) have the different type of singularity — the "bolt"-type — when only 1-dimensional circle in the Hopf bundle of S^3 is collapsed. To describe such metrics one is led to the flows with much more unpleasant right-hand side. For example, if the metric $g(t)$ satisfies the flow

$$\frac{\partial}{\partial t} g = \frac{1}{2} \sqrt{\det(Ric)} Ric^{-1}.$$

then the metric $\bar{g}$ will be Eguchi-Hanson metric for appropriate initial data.

Discrete Dynamics of the Tyurin Parameters and Commuting Difference Operators

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We study commuting difference operators of rank two. In the case of hyperelliptic spectral curves an equation which is equivalent to the Krichever – Novikov equations on Tyurin parameters is obtained. With the help of this equation examples of operators corresponding to hyperelliptic spectral curves of arbitrary genus are constructed. Among these examples there are operators with polynomial and trigonometric coefficients.

If two difference operators

$$L_4 = \sum_{i=-2}^{2} u_i(n)T^i, \qquad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n)T^i, \qquad u_2 = v_{2g+1} = 1$$

commute, where T — shift operator, then there is a nonzero polynomial $F(z,w)$ such that $F(L_4, L_{4g+2}) = 0$. The polynomial F defines the *spectral curve* of L_4, L_{4g+2}

$$\Gamma = \{(z,w) \in \mathbb{C}^2 | F(z,w) = 0\}.$$

The common eigenvalues are parametrized by the spectral curve

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, (z,w) \in \Gamma.$$

The *rank* of the pair L_4, L_{4g+2} is called the dimension of the space of common eigenfunctions for fixed eigenvalues

$$l = \dim\{\psi\colon L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad (z,w) \in \Gamma.\}$$

The curve Γ admits a holomorphic involution

$$\sigma\colon \Gamma \to \Gamma, \quad \sigma(z,w) = \sigma(z,-w).$$

The common eigenfunctions L_4 and L_{4g+2} satisfy the equation

$$\psi_{n+1}(P) = \chi_1(n,P)\psi_{n-1}(P) + \chi_2(n,P)\psi_n(P),$$

The functions $\chi_1(n,P)$ and $\chi_2(n,P)$ are rational on Γ and have $2g$ simple poles depending on n. In addition the function $\chi_2(n,P)$ has a simple pole in q. For finding L_4 and L_{4g+2} it is sufficient to find χ_1 and χ_2.

The following theorems are proved.

Theorem 1. *If*

$$\chi_1(n,P) = \chi_1(n,\sigma(P)), \qquad \chi_2(n,P) = -\chi_2(n,\sigma(P)),$$

then L_4 has the form

$$L_4 = (T + V_nT^{-1})^2 + W_n,$$

herewith

$$\chi_1 = -V_n \frac{Q_{n+1}}{Q_n}, \qquad \chi_2 = \frac{w}{Q_n},$$

where

$$Q_n(z) = z^g + \alpha_{g-1}(n) z^{g-1} + \ldots + \alpha_0(n).$$

Functions V_n, W_n, Q_n *satisfy the following equation*

$$F_g(z) = Q_{n-1} Q_{n+1} V_n + Q_n (Q_{n+2} V_{n+1} + Q_{n+1}(z - V_n - V_{n+1} - W_n)).$$

Theorem 2. *The operator*

$$L_4 = (T + (r_3 n^3 + r_2 n^2 + r_1 n + r_0) T^{-1})^2 + g(g+1) r_3 n$$

commutes with a difference operator L_{4g+2} *of order* $4g+2$, *where* r_0, r_1, r_2, r_3 *— parameters,* $r_3 \neq 0$.

Theorem 3. *The operator*

$$L_4 = (T + (r_1 a^n + r_0) T^{-1})^2 + (a^{2g+1} - a^{g+1} - a^g + 1) r_1 a^{n-g}$$

commutes with a difference operator L_{4g+2}, *where* r_0, r_1, a *are parameters such that* $r_1 \neq 0$, $a \neq 0$, $a^{2g+1} - a^{g+1} - a^g + 1 \neq 0$.

Theorem 4. *The operator*

$$L_4 = (T + (r_1 \cos(n) + r_0) T^{-1})^2 - 4 r_1 \sin(\frac{g}{2}) \sin(\frac{g+1}{2}) \cos(n + \frac{1}{2})$$

commutes with a difference operator L_{4g+2}, *where* r_0, r_1 *— parameters,* $r_1 \neq 0$.

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The Spectrum of the Curvature Operators of the Conformally Flat Metric Lie Groups

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The report examines the spectra of the sectional curvature, Ricci curvature, one-dimensional curvature operators of the conformally flat metric Lie groups. We discuss some examples of such groups.

The Heat Kernel and Its Asymptotic on the Diagonal for an Optimal Control Problem with Drift

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In this talk we will consider an optimal control problem defined on the n dimensional Euclidean space depending linearly on $k \leq n$ controls, with a drift vector field and a quadratic cost. We will introduce a related hypoelliptic differential operator, being interesteded in the fundamental solution and its asymptotic expansion on the diagonal for small time. In particular, in the linear case we will show the explicite solution and compute the first terms of the asymptotic. We will then use these results to investigate the general case, and we will show the first terms of the asymptotic in some cases.

Some Topics in Modern Quantum Control

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Control of atomic and molecular scale systems with quantum dynamics attracts nowadays high interest due to rich mathematical theory and various existing and prospective applications in physics, chemistry, and molecular biology including laser-assisted control of chemical reactions, quantum metrology, quantum optics, etc. Modern quantum technologies which might revolutionize our society like semiconductor revolution did in the second half of the twentieth century, are based on methods of quantum control [1–4].

Mathematical formulation of a quantum control problem included description of state space of the system, the dynamical equation, and specification of the target objective functional. The dynamics of the controlled quantum system is governed either by Schrödinger equation if the system is closed, that is, isolated from the environment, or by a master-equation if the system is open, that is, interacts with an environment. In both cases the evolution equation includes the control function which can be shaped laser field, spectral density of incoherent photons, or other external action. Objective functional can describe probability of transition from one state to another, average value of quantum observable, gate generation, etc. The goal of the optimal control is to find such a control function which maximizes the objective functional.

In this talk we will discuss recent progress in two very important and interesting topics in modern quantum control—controllability of open quantum systems and the analysis of quantum control landscapes.

Controllability of quantum systems deals with finding methods for transferring arbitrary initial states into arbitrary final states with admissible controls. We will discuss a method for a controlled engineering of arbitrary quantum states (density matrices) of n-level quantum systems which might be used for prospective quantum computing with mixed states [5].

Analysis of the control landscape, that is, graph of the objective functional, deals with the analysis of local but not global extrema (traps) of the objective functional. We will discuss the recent discovery of absence of traps for two-level systems [6,7] which are important as representing qubit—a basis building block for quantum computation, and for systems with infinite-dimensional state space, namely, for transmission coefficient of a quantum particle on the line passing through one-dimensional potential whose shape is used as a control [5]. For the latter, we consider a quantum particle of energy E moving from the left in one dimensional potential $V(x)$ which is assumed to have compact support. Probability for the particle to appear far away on the right of the potential is the transmission coefficient $T_E[V]$. The transmission coefficient is a functional of the potential $V(x)$ and can be controlled by varying its shape. We show that the only extrema of the transmission coefficient as a functional of the potential V are global maxima corresponding to full transmission [8]. This result is of high mathematical importance as the first result about absence of traps for quantum systems with infinite dimensional state space and of high practical significance as it says that manipulating by transmission coefficient is trap free.

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Optimal Quantum Control of the Landau–Zener System by Measurements

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In the recent works by A. Pechen et al. and F. Shuang et al. a problem of optimal control of a two-level quantum system by nonselective measurements was considered. In these works, the time instants of measurements are fixed; the maximization of a transition probability is performed over various observables. Note that, in case of two-level system, quantum dynamics without measurements is a unitary evolution in the two-dimensional complex vector space; a von Neumann observable is specified by a unit vector of the space.

In the present work, we consider a special (but important) case of two-level quantum system, namely, the Landau–Zener system (spin-1/2 charged particle in time-dependent magnetic field). We consider a problem of maximization of a transition probability when an observable is fixed, but instants of measurements are variable. We obtain full exact solution of the maximization problem in the large coupling constant limit for an arbitrary number of measurements. Also we establish a duality between two different problem statements: maximization over various observables under fixed time instants of measurements and maximization over various time instants under a fixed observable.

Cut Locus in the Riemannian Problem on SO_3 in Axisymmetric Case

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The parametrization of Riemannian geodesics on SO_3 is the classic L. Euler's result. But the global optimality of geodesics was not investigated yet. L. Bates and F. Fasso [1] have got the equation for conjugate time in the axisymmetric case and described conjugate locus depending on the ratio of eigenvalues of Riemannian metric.

We represent the Maxwell strata, cut locus and give the equation for the cut time.

When one eigenvalue of Riemannian metric moves to infinity, the parametrization of geodesics, conjugate time and locus, cut time and locus in the Riemannian problem converge to the sub-Riemannian ones that were considered by U. Boscain and F. Rossi [2].

Let $I_1 = I_2, I_3$ be the eigenvalues of the left invariant Riemannian metric, e_1, e_2, e_3 be corresponding basis in $\mathfrak{so}_3$, and p_1, p_2, p_3 be corresponding impulses, $\bar{p}_i = \frac{p_i}{|p|},\ i = 1, 2, 3$.

Let $\eta = \frac{I_1}{I_3} - 1 > -1$.

Theorem. *Let $\tau_{cut}(\eta, \bar{p}_3)$ be the minimal positive root of the equation*

$$\cos\tau\cos(\tau\eta\bar{p}_3) - \bar{p}_3\sin\tau\sin(\tau\eta\bar{p}_3) = 0$$

(1) If $\eta \geqslant -\frac{1}{2}$, then the cut time is $\frac{2I_1\tau_{cut}(\eta,\bar{p}_3)}{|p|}$.
(2) If $\eta < -\frac{1}{2}$, then the cut time is

$$\begin{cases} \frac{2\pi I_1}{|p|}, & \text{if } \frac{1}{2\eta} \leqslant |\bar{p}_3| < 1, \\ \frac{2I_1\tau_{cut}(\eta,\bar{p}_3)}{|p|}, & \text{if } |\bar{p}_3| < \frac{1}{2\eta}. \end{cases}$$

Theorem. *(1) If $\eta \geqslant -\frac{1}{2}$, then the cut locus is $\mathbb{R}P^2$ consisting of rotations by angles π in SO_3.*
(2) If $\eta < -\frac{1}{2}$, then the cut locus contains two components: $\mathbb{R}P^2$ and the segment

$$J_\eta = \{\exp(\pm\varphi e_3) \mid \varphi \in [2\pi(1+\eta), \pi]\}.$$

The proof of these theorems is based on considering the symmetry group of Hamiltonian vector field of Pontryagin maximum principle, finding its fixed points (Maxwell strata), considering some open sets bounded by Maxwell strata, which are diffeomorphic by exponential map. This method was presented by Yu. L. Sachkov for the Euler elastic problem [3].

Proposition. *The geodesics parametrization, conjugate time and locus, cut time and locus in sub-Riemannian problem on SO_3 are obtained from the Riemannian ones by $I_3 \to \infty$.*

Example. Cut locus in the sub-Riemannin problem has two components: $\mathbb{R}P^2$ and the circle without point

$$S^1 \setminus \{\mathrm{id}\} = \{\exp(\varphi e_3) \mid \varphi \in (0, 2\pi)\}.$$

The stratum J_η of the cut locus for the Riemannian problem converges to this circle without point if $\eta \to -1$ (equivalent to $I_3 \to \infty$).

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The Laplace–Beltrami Operator on Conic and Anti-conic Surfaces

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We consider the evolution of a free particle on a two-dimensional manifold endowed with the degenerate Riemannian metric $ds^2 = dx^2 + |x|^{2\alpha} d\theta^2$, where $x \in R$, $\theta \in S^1$ and the parameter $\alpha \in R$. For α smaller or equal to -1 this metric describes cone-like manifolds (for $\alpha = -1$ it is a flat cone). For $\alpha = 0$ it is a cylinder. For α bigger or equal to 1 it is a Grushin-like metric.

In particular, we discuss whether a free particle or the heat can cross the singular set $x = 0$ or not, and in which cases the singularity absorbs the heat. (The latter problem is known as the stochastic completeness problem.)

In the last part of the talk we will present some recent results regarding the spectrum of the Laplace–Beltrami operator associated with these metrics and the Aharonov-Bohm effect in the Grushin case.

This is a joint work with U. Boscain and M. Seri.

Comparison Theorems in Sub-Riemannian Geometry

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The typical Riemannian comparison theorem is a result in which a *local* bound on the curvature (e.g. Ric $\geq \kappa$) implies a *global* comparison between some property on the actual manifold (e.g. diameter) and the same property on a constant curvature model. The generalization of these results to the sub-Riemannian setting is not straightforward, the main difficulty being the lack of a proper theory of Jacobi fields, an *analytic* definition of curvature and, a fortiori, constant curvature models.

Some comparison results, valid for 3D sub-Riemannian structures, have been recently obtained by Agrachev and Lee and generalized to contact manifolds with symmetries by Lee, Li and Zelenko. Building on these results, we develop a theory of Jacobi fields valid for *any* sub-Riemannian manifold, in which the Riemannian *sectional* curvature is generalized by the *canonical curvature* introduced by Agrachev and his students.

This allows to extend a wide range of comparison theorems to the sub-Riemannian setting. In particular, we focus on sectional and Ricci-type comparison theorems for the existence of conjugate points along sub-Riemannian geodesics. In this setting, the models with constant curvature are represented by Linear-Quadratic optimal control problems with constant potential. As an application, we prove a sub-Riemannian version of the Bonnet-Myers theorem and we obtain some new results on conjugate points for three dimensional left-invariant sub-Riemannian structures.

This is a joint work with D. Barilari (Paris 7).

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Finite-Gap 2D-Schrödinger Operators with Elliptic Coefficient

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In general case the potential of the finite-gap Schrödinger operator $-\frac{\partial^2}{\partial x^2}+u(x)$ is expressed in terms of theta function of the spectral curve [3]. At the same time there are examples of finite-gap operators with elliptic potentials, for example, the Lamé operators $-\frac{\partial^2}{\partial x^2}+g(g+1)\wp(x)$ or the Treibich-Verdier operator $-\frac{\partial^2}{\partial x^2}+\sum_{i=0}^{3}a_i(a_i+1)\wp(x+\omega_i)$, where ω_i are semi-periods. Theorems 1 and 2 show that the same phenomena are possible in two-dimensional case.

Theorem 1. The Schrödinger operator

$$H=\frac{\partial^2}{\partial z\partial \bar z}+a\left(\frac{\sqrt{g_0}-\wp'(az+b\bar z)}{2\wp(az+b\bar z)}\right)\frac{\partial}{\partial \bar z}-\frac{bg(g+1)\wp(az+b\bar z)}{2a} \tag{1}$$

is finite-gap, where $\wp$ is elliptic Weierstrass function satisfying the equation

$$(\wp'(z))^2=\frac{2g(g+1)}{a^2}\wp(z)^3+g_2\wp(z)^2+g_1\wp(z)+g_0.$$

The spectral curve of the operator H is a hyperelliptic curve with genus g.

Thus for the operator H theta functional formulas for the coefficients is reduced to the simpler formulas (1). Note that H satisfies the identity

$$\left[H,-\frac{\partial^2}{\partial z^2}+g(g+1)\wp(az+b\bar z)\right]=-2a\left(\frac{\partial}{\partial z}\left(\frac{\sqrt{g_0}-\wp'(az+b\bar z)}{2\wp(az+b\bar z)}\right)\right)H.$$

Theorem 2. The Schrödinger operator

$$H=\frac{\partial^2}{\partial z\partial \bar z}+\frac{7a\wp'(az+b\bar z)}{20g_2a^2-14\wp(az+b\bar z)}\frac{\partial}{\partial \bar z}+\frac{b\wp(az+b\bar z)}{2a}$$

is finite-gap, where $\wp$ is elliptic Weierstrass function satisfying the equation

$$(\wp'(z))^2=-\frac{1}{2a^2}\wp(z)^3+g_2\wp(z)^2-\left(\frac{7g_0}{10g_2a^2}+\frac{20g_2^2a^2}{49}\right)\wp(z)+g_0.$$

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Local and Metric Geometry of Nonregular Weighted Carnot–Carathéodory Spaces

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We investigate local and metric geometry of weighted Carnot–Carathéodory spaces in a neighbourhood of a nonregular point [8]. Such spaces are a wide generalization of classical sub-Riemannian spaces (which are smooth manifolds equipped by bracket-generating distributions of "horizontal" vector fields) and naturally arise in control theory (including cases when the dependence on control functions may be nonlinear), harmonic analysis, subelliptic equations etc.

For the spaces that we consider, there may be no analog of the intrinsic Carnot–Carathéodory metric (defined in sub-Riemannian geometry as the infimum of lengths of all "horizontal" curves joining the two given points) might not exist, and some other new effects, caused by the arbitrary weights of the vector fields, take place, which leads to necessity of introducing new methods of investigation of geometry of such spaces.

We describe the local algebraic structure of such a space, endowed with a natural quasimetric (first introduced by A. Nagel, E. M. Stein and S. Wainger in [5]) induced by the given weighted structure. We compare local geometries of the initial CC space and its tangent cone (which is a homogeneous space of a nilpotent Lie group) at some fixed (maybe nonregular) point.

Our considerations heavily rely on similar results about equiregular Carnot–Carathéodory spaces [4, 3] and adaptations of different "lifting" methods [6, 2, 1], which allow to reduce some questions about nonregular spaces to similar questions about the equiregular ones. Also, we use a generalisation to quasimetric spaces of the Gromov–Hausdorff spaces for metric spaces, which was constructed earlier in [7], and study new properties of the considered quasimetrics.

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On Conjugate Times of LQ Optimal Control Problems

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We consider an LQ optimal control problem, more generally a dynamical system with a constant quadratic Hamiltonian, and we characterize the number of conjugate times in terms of the spectrum of the Hamiltonian vector field $\vec{H}$. We prove the following dichotomy: the number of conjugate times is identically zero or grows to infinity. The latter case occurs if and only if $\vec{H}$ has at least one Jordan block of odd dimension corresponding to a purely imaginary eigenvalue. As a byproduct, we obtain bounds from below on the number of conjugate times contained in an interval in terms of the spectrum of $\vec{H}$.

Theorem. *The conjugate times of a controllable linear quadratic optimal control problem obey the following dichotomy:*

- *If the Hamiltonian field $\vec{H}$ has at least one odd-dimensional Jordan block corresponding to a pure imaginary eigenvalue, the number of conjugate times in the interval $[0, T]$ grows to infinity for $T \to \pm\infty$.*
- *If the Hamiltonian field $\vec{H}$ has no odd-dimensional Jordan blocks corresponding to a pure imaginary eigenvalue, there are no conjugate times.*

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Analytical Properties of Sobolev Mappings on Roto-Translation Groups

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The roto-translation group, $SE(2)$, is a three-dimensional topological manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ with coordinates (x, y, θ). The left-invariant vector fields

$$X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta},\ X_3 = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y},$$

form a basis of the Lie algebra of $SE(2)$. The bracket-generating subbundle of the tangent-bundle is spanned by the frame X_1, X_2.

Consider the basic 1-forms dX_1, dX_2, dX_3, dual to the basic vector fields X_1, X_2, X_3, i.e., $dX_i(X_j) = \delta_{ij}$. Applying the methods developed in [1] we establish a key relation underlying the connection between mappings with bounded distortion [2] and nonlinear potential theory.

Theorem. *Let $SE(2)$ be a roto-translation group and $\Omega \subset SE(2)$ is an open set. Suppose that $f\colon \Omega \to SE(2)$ is a Sobolev mapping of the class $W^1_{4,\mathrm{loc}}(\Omega)$, $V\colon SE(2) \to \mathbb{R}^2$ is a vector field $V = (v_1, v_2) \in C^1$ such that $\mathrm{div}_h V = X_1 v_1 + X_2 v_2$ is bounded on $SE(2)$, and*

$$\omega(g) = v_1(g)\, dX_2 \wedge dX_3 - v_2(g)\, dX_1 \wedge dX_3, \quad g \in \Omega.$$

Then the equality $df^{\#}\omega = f^{\#} d\omega$ holds in the sense of distributions.

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